Sample Questions for KAUST Mathematics Competition
Final Round, Seniors Track

1. What is the least positive integer n such that $n \cdot 1! \cdot 4! \cdot 4! \cdot 7!$ is a perfect square?
Note: For a positive integer k, $k!$ is the product of all positive integers from 1 to k. That is, $k! = k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1$.

<u>**Sol 1.**</u> Notice that the expression is equivalent to: $n \cdot (4!)^2 \cdot 2^4 \cdot 3^2 \cdot 5 \cdot 7$. Since $(4!)^2 \cdot 2^4 \cdot 3^2$ is a perfect square, $35n$ must be a perfect square meaning that $n = 35$ is the minimum possible

2. Hatem selects three different integers at random from the set $\{-5, -4, -3, \cdots, 4, 5\}$. What is the probability that the sum of the selected numbers is 0?

<u>**Sol 2.**</u> To find the probability, we divide the number of successful outcomes (unordered triples that sum to zero) by the total number of possible outcomes.
1. Total Number of Outcomes: The set contains 11 integers. Since the order of selection does not matter and the integers must be distinct, the total number of possible triples is given by the combination formula $\binom{n}{k}$: $\textbf{Total} = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} = 165$.
2. Successful Outcomes (Sum = 0): We seek triples $\{a, b, c\}$ such that $a + b + c = 0$. To ensure each triple is unique and avoid double-counting, we assume $a < b < c$. We organize the count based on the largest integer c:

- Case $c = 5$: $\{-5, 0, 5\}, \{-4, -1, 5\}, \{-3, -2, 5\}$ (**3 triples**)
- Case $c = 4$: $\{-5, 1, 4\}, \{-4, 0, 4\}, \{-3, -1, 4\}$ (**3 triples**)
  (*Note:* $\{-2, -2, 4\}$ is excluded because the integers must be distinct.)
- Case $c = 3$: $\{-5, 2, 3\}, \{-4, 1, 3\}, \{-3, 0, 3\}, \{-2, -1, 3\}$ (**4 triples**)
- Case $c = 2$: $\{-3, 1, 2\}, \{-2, 0, 2\}$ (**2 triples**).
- Case $c = 1$: $\{-1, 0, 1\}$ (**1 triple**)

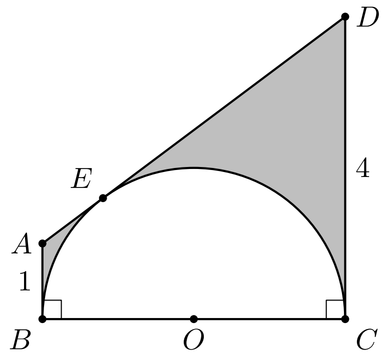
Summing the successful cases: $3 + 3 + 4 + 2 + 1 = 13$. The desired probability is $\frac{13}{165}$.

3. Find all triples x, y, z satisfying both equations

$$x^2 + 4y^2 + 2(x - 2y) + 2 = 0 \quad \text{and} \quad 2xy + 4yz + zx - 4 = 0.$$

<u>**Sol 3.**</u> The first equation can be written as $(x+1)^2 + (2y-1)^2 = 0$. This means that $x = -1$ and $y = \frac{1}{2}$. Substituting in the second equation, we get $-1 + 2z - z - 4 = 0$, thus $z = 5$. Hence, $(x, y, z) = \left(-1, \frac{1}{2}, 5\right)$.

4. In the diagram below, $ABCD$ is a trapezoid with $AB \parallel CD$, $BC \perp AB$, $AB = 1$, and $CD = 4$. A semicircle with center O and diameter $\overline{BC}$ is tangent to $\overline{AD}$ at E. Calculate the area of the shaded region.



Sol 4. Solution 1. Let $\mathcal{O}$ denote the circle. Since AD is tangent to the circle $\mathcal{O}$ at E, we have AO, DO bisect angles $\angle EAB, \angle EDC$ respectively, which gives $\angle AOD = 90°$ since $AB||CD$. Therefore, $\triangle ABO \sim \triangle OCD$ and we get the ratios

$$\frac{AB}{OC} = \frac{BO}{CD}.$$

From that, we obtain $OC \cdot OB = OB^2 = AB \cdot CD = 4 \Rightarrow OC = 2$. Therefore, the area of the shaded diagram is the area of the trapezoid minus the area of the half circle with radius $OC = 2$. That is,

$$\text{Area}(ABCD) - \frac{\text{Area}(\mathcal{O})}{2} = \frac{AB + CD}{2} \cdot BC - \frac{\pi OC^2}{2} = \frac{5}{2} \cdot 4 - \frac{4\pi}{2} = 10 - 2\pi.$$

Solution 2. Since $AE = AB = 1$ and $DE = DC = 4$, we have $AD = AE + ED = 5$. To find BC, we use Pythagorean Theorem in $\triangle ADF$, where F is a feet of A to CD. Hence, $BC = \sqrt{AD^2 - DE^2} = \sqrt{5^2 - 3^2} = 4$. Therefore, the area of shaded region is the area of trapezoid minus the area of the semicircle, i.e $\frac{(1+4)\cdot 4}{2} - \frac{\pi \cdot 2^2}{2} = 10 - 2\pi$.

5. Determine the largest possible integer k such that $300^9 - 3^9$ is divisible by 3^k.

Sol 5. Solution 1. Notice that $300^9 - 3^9 = 3^9(100^9 - 1) = 3^9(10^{18} - 1) = 3^9(10^9 - 1)(10^9 + 1)$. The number $10^9 + 1$ is not divisible by 3 while $10^9 - 1$ is. Further

$$10^9 - 1 = (10^3 - 1)(10^6 + 10^3 + 1)$$

Notice that $10^3 - 1 = 27 \cdot 37$ and the number of $10^6 + 10^3 + 1$ is divisible by 3 but not 9 since the sum of its digits is equal to 3. Thus, $10^9 - 1$ would be divisible by 3^4 and won't be divisible by 3^5. Thus, $k = 4 + 9 = 13$.

Solution 2. Writing $300^9 - 3^9$ in the form,

$$3^9(10^{18} - 1) = 3^9 \cdot \underbrace{99\ldots9}_{18} = 3^{11} \cdot \underbrace{11\ldots1}_{18},$$

by long division, we find

$$3^9(10^{18} - 1) = 3^{13} \cdot 12345679012345679.$$

Thus, we get that $k = 13$, because the second factor is not multiple of 3 as its sum of digits is equal to 74.

6. Four polygons – one equilateral triangle and three other congruent regular polygons, all with unit side length – are placed on a table. Every two polygons share exactly one side, and no two overlap. What is the perimeter of the shape formed by these polygons?

Sol 6. Let the three congruent polygons each have n sides. Since each polygon shares one side with the equilateral triangle, the three polygons are arranged around the triangle. Therefore, at each vertex of the triangle, two of the polygons meet together with the triangle. Hence, these two polygons share one side. So, at each vertex of a triangle, we have three figures – one equilateral triangle and two polygons – with sum of interior angles equal to 360°. Since these angles are 60° and two others are $\frac{(n-2)180^\circ}{n}$ (angle of $n-$gon), we get the following equation,

$$60^\circ + \frac{2 \cdot (n-2) \cdot 180^\circ}{n} = 360^\circ.$$

Solving this equation, we get $n = 12$. Therefore, the perimeter of the resulting shape is $3(n-3) = 27$.

7. Let $x_1, x_2, x_3, \cdots, x_{2026}$ be a sequence of real numbers that satisfy the relation:

$$x_{i+1} = \frac{i - x_i}{i}$$

for all $i = 1, 2, \ldots, 2025$. Find the sum:

$$S = x_1 + 2x_2 + 3x_3 + \cdots + 2024x_{2024} + 2025x_{2025} + 2025x_{2026}.$$

Sol 7. Notice that the relation is equivalent to

$$ix_{i+1} = i - x_i \iff (i+1)x_{i+1} = (x_{i+1} - x_i) + i, \quad i \geq 1.$$

Now we sum this for all i up to 2025 to get:

$$\sum_{i=1}^{2025} (i+1)x_{i+1} = \sum_{i=1}^{2025} (x_{i+1} - x_i) + \sum_{i=1}^{2025} i$$

Simplifying:

$$S + x_{2026} - x_1 = x_{2026} - x_1 + \frac{2025 \cdot 2026}{2} \Rightarrow S = \frac{2025 \cdot 2026}{2}.$$

8. Several distinct integers, each greater than 20, are written on a board. The product of the smallest and largest of them equals the sum of the remaining numbers. What is the minimum possible number of integers on the board?

Sol 8. Let $20 < x_1 < x_2 < \cdots < x_n$ be the numbers on the board. Then they satisfy

$$x_1 x_n = x_2 + x_3 + \ldots + x_{n-1}.$$

The left side of this equation is at least $21x_n$, but the right side is less than $(n-2)x_n$. These imply $21 < n - 2$ or equivalently $n \geq 24$. To show that $n = 24$ is attainable, we consider the following 24 integers, all greater than 20

$$21,\ 231,\ 232,\ \ldots,\ 252,\ 253.$$

The sum of the 22 (except the first and the last) numbers is equal to

$$231 + 232 + \cdots + 252 = \frac{(231 + 252) \cdot 22}{2} = 483 \cdot 11 = 5313,$$

which is $x_1 \cdot x_n = 21 \cdot 253$.