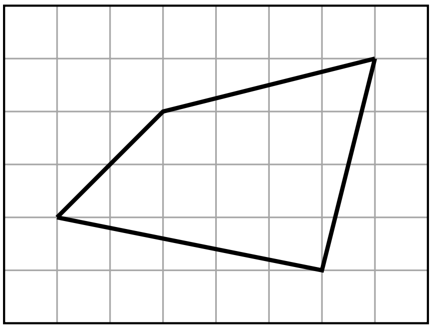
Sample Questions for KAUST Mathematics Competition
Final Round, Juniors Track

1. In an arithmetic sequence (one in which the difference between consecutive terms is constant), five consecutive terms are $a, x, b, c, 2x$. What is the value of $\frac{a+b}{c}$?

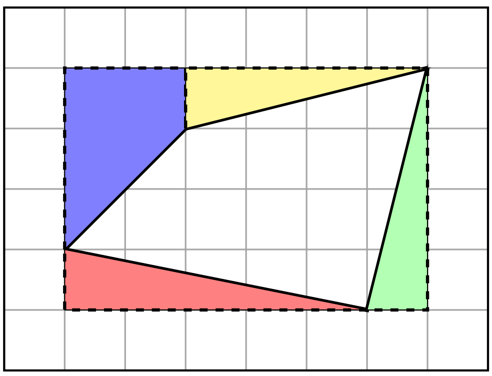
Sol 1. Notice that $2x = a + b$, $2c = b + 2x$, $2b = c + x$. Eliminating b from the last two equations, it follows that $c = \frac{5x}{3}$. Hence, the desired value is $\frac{6}{5}$.

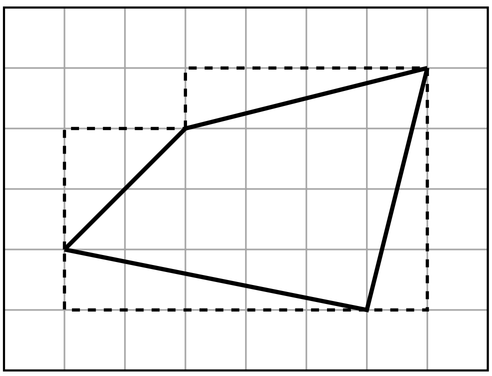
2. Find the area and the perimeter of the quadrilateral shown below, assuming each small square has side length 1, and the vertices of the quadrilateral are on grid points.



Sol 2. The area of the quadrilateral equals the area of the 4×6 dashed rectangle minus the areas of the colored regions (see the left figure). The area of the red region is $5/2 = 2.5$, the green region is 2, the blue is 4, and the yellow is 2. Therefore, $4 \cdot 6 - (2 + 4 + 2 + 2.5) = 13.5$. To find the perimeter, we use the right triangles formed by the dashed legs (see the right figure). Applying the Pythagorean theorem to each triangle, we obtain

$$\sqrt{4+4} + \sqrt{1+16} + \sqrt{1+16} + \sqrt{1+25} = 2\sqrt{2} + 2\sqrt{17} + \sqrt{26}.$$





3. When 95 is divided by a positive integer N, the remainder is 4. What is the least possible value of N?

Sol 3. As N is greater than 4 and divides $95-4=91=7\cdot 13$, the smallest possible value is 7.

4. All positive even integers are written consecutively:

$$246810121416182022\ldots$$

Which digit appears in the 2026-th position?

Sol 4. 1-digit even numbers are $2,4,6,8$, so 4 digits in total.
2-digit even numbers are $10,12,\ldots,98$, so 45 numbers, contributing $45\cdot 2=90$ digits.
3-digit even numbers are $100,102,\ldots,998$. There are 450 numbers, contributing $450\cdot 3=1350$ digits.
Thus, the 2026-th digit lies among the 4-digit even numbers. Its position within these numbers is

$$2026-(4+90+1350)=582.$$

Since $582=4\cdot 145+2$ and the first 145 4-digit even numbers are $1000,1002,\ldots,1288$, the answer is the second digit of 1290, i.e. it is 2.

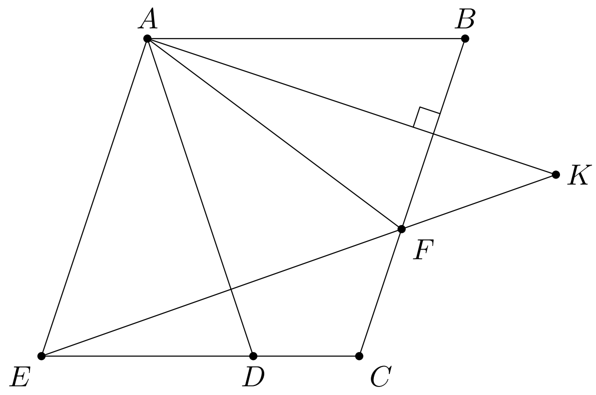
5. Find all values of a,b and c that satisfy the system of equations:

$$\begin{cases} ab-a-b=-7 \\ bc-b-c=7 \\ ca-c-a=-13. \end{cases}$$

Sol 5. Solution 1. By adding 1 to the first equation, we obtain: $ab-a-b+1=-6 \Rightarrow (a-1)(b-1)=-6$. By doing the same for the other equations and multiplying these, we get: $(a-1)(b-1)(c-1)=\pm\sqrt{(-6)(8)(-12)}=\pm 24$.
Case 1 (+24): Dividing by $(a-1)(b-1)=-6$, we get: $c-1=-4$. Analogously, $a-1=3$ and $b-1=-2$. Therefore $a=4, b=-1, c=-3$.
Case 2 (-24): Similarly as the previous case, we get $c-1=4, a-1=-3, b-1=2 \Rightarrow a=-2, b=3, c=5$.

Solution 2. We express the first equation as $a(b-1)=(b-7)$. Now $(b-1)$ is clearly nonzero, hence we divide and get $a=(b-7)/(b-1)$. Substituting this into the third equation, we arrive at $c=2b-1$. Substituting this into the second equation gives $b^2-2b-3=0$ which implies $b=-1$ or $b=3$. Solving backwards, $(a,b,c)=(-2,3,5)$ or $(4,-1,-3)$.

6. In the diagram below, $ABCD$ is an isosceles trapezoid, $ABCE$ is a rhombus, and $\angle ABC=72^\circ$. The line AF bisects $\angle DAB$, and K is the point on EF such that $AK\perp BC$. Find $\angle AKE$.



Sol 6. Since $ABCD$ is an isosceles trapezoid, $\angle BAD = 72°$. Therefore, $\angle BAF = 36°$. Considering the triangle ABF, we find $\angle AFB = 72°$, thus $AF = AB$. Since $ABCE$ is a rhombus, this means $AF = AE$. On the other hand, $AE \parallel BC$, thus $\angle EAF = \angle AFB = 72°$, hence $\angle AFE = \angle AEF = 54°$. Finally, considering triangle ΔEAK and the fact that $\angle KAE = 90°$, we find $\angle AKE = 36°$.

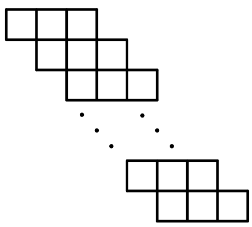
7. Find all primes p, q, r and s (not necessarily distinct) that satisfy

$$p \cdot q \cdot r \cdot s = (p+1)(q+1)(r+1).$$

Sol 7. Without loss of generality, assume $r = \max(p, q, r)$. Since r does not divide $r+1$, the condition r divides $(p+1)(q+1)(r+1)$ implies that r divides $p+1$ or $q+1$. Again without loss of generality, suppose r divides $p+1$. Since $r \geq p$, we must have $r = p+1$, which forces $p = 2$ and $r = 3$. Now $q \leq r = 3$ restricts q to 2 or 3. A direct check shows that $q = 3$ does not yield an integer value for s, whereas $q = 2$ gives $s = 3$. Therefore the solutions are all permutations of (p, q, r) with $s = 3$:

$$(p,\ q,\ r,\ s)\ \in\ \{(2,2,3,3),\ (2,3,2,3),\ (3,2,2,3)\}.$$

8. A stair shape consists of 100 rows, each containing three squares, where each row is shifted one square to the right relative to the row above it, as shown below. In how many ways can the numbers 1 to 300 be placed in the squares (one number per square, each used exactly once) so that the numbers increase from left to right in each row and from top to bottom in each column?



Sol 8. Consider the row number k and consider the middle square. The number in this square is greater than that in the left square as well as all the numbers in the rows above, which are in total $1 + 3(k-1) = 3k-2$ numbers. On the other hand, it is less than the number in the right square as well as all the numbers in the rows below, which are in total $1+3(100-k) = 301-3k$ numbers. Therefore, the number in this square is forced to be $3k-1$, filling all the middle squares from the top to the bottom as $2, 5, 8, 11, \ldots$. This also forces the left square of the top row to be 1, and the right square of the bottom row to be 300. For the other squares, we consider the right square of row k together with the left square of row $(k+1)$ and notice that they are both between the middle squares of rows k and $(k+1)$, which are $(3k-1)$ and $(3k+2)$. Hence these two squares must be filled with $3k$ and $3k+1$, but either way of filling is valid. Considering $k = 1, 2, \ldots, 99$, we cover all the squares, hence see that the total number of ways to fill the squares is 2^{99}.